Small Examples of Non-Constructible Simplicial Balls and Spheres

Frank H. Lutz

Abstract

We construct non-constructible simplicial d-spheres with $d+10$ vertices and non-constructible, non-realizable simplicial d-balls with $d+9$ vertices for $d \geq 3$.

1 Introduction

The concepts of *vertex-decomposability*, *shellability*, and *constructibility* describe three particular ways to assemble a simplicial complex from the collection of its facets (cf. Björner [4]). The following implications are strict for (pure) simplicial complexes:

$$\text{vertex decomposable} \Longrightarrow \text{shellable} \Longrightarrow \text{constructible.}$$

Shellability has its origin in Schläfli's computation from 1852 [32] of the Euler characteristics of convex polytopes, where he based his calculation on the assumption that the boundary complexes of polytopes are shellable. However, this property of polytopes was justified only much later in 1970 by Bruggesser and Mani [8] and then played a crucial role in McMullen's proof of the Upper Bound Theorem in the same year [27]. Besides in polyhedral theory, shellability has found fruitful applications in topology, combinatorics, and computational geometry; see the surveys [3], [4], [11], [34, Ch. 8], [35], and the references contained therein.

The notion of constructibility was coined by Hochster in 1972 [18], but implicitly was used long before in combinatorial topology. In particular, it follows from Newman's and Alexander's fundamental works on the foundations of combinatorial and PL topology from 1926 [28] and 1930 [1] (cf. also Björner [4]) that a constructible d-dimensional simplicial complex in which every $(d-1)$-face is contained in exactly two or at most two d-dimensional facets is a PL d-sphere or a PL d-ball, respectively. For recent surveys on constructibility see [16] and [17].

The strongest concept, vertex-decomposability, was introduced by Provan and Billera in their proof from 1980 [30] that vertex decomposable simplicial complexes satisfy the simplicial form of the famous Hirsch conjecture (cf. [12, p. 168]) of linear programming.

Although boundary spheres of simplicial polytopes are shellable, Lockeberg [23] constructed a simplicial 4-polytope with 12 vertices that is not vertex-decomposable; and there even are not vertex-decomposable simplicial 4-polytopes with 10 vertices [20] and not vertex-decomposable, non-polytopal simplicial 3-spheres with 9 vertices [7]. For two-dimensional balls and spheres it was proved by Bing [3] that they are shellable and by Provan and Billera [30] that they are vertex-decomposable. Klee and Kleinschmidt [20] also showed that all simplicial d-balls and all simplicial d-spheres with up to $d+3$, respectively $d+4$ vertices, are vertex-decomposable. However, for $d \geq 3$ there are not vertex-decomposable simplicial d-balls with $d+4$ vertices and 10 facets as well as not vertex-decomposable simplicial d-spheres with $d+6$ vertices; see [7] and [26].

The first known example of a non-shellable cellular 3-ball is due to Furch and appeared in 1924 [14]. A non-shellable simplicial 3-ball with 30 vertices and 72 facets was provided by Newman in 1926 [29]. Newman's ball is *strongly non-shellable*, i.e., it has no *free* facet that can be removed from the triangulation without loosing ballness. Much smaller strongly non-shellable simplicial 3-balls were obtained by Grünbaum (cf. [11]) with 14 vertices and 29 facets and by Ziegler [35] with 10 vertices and 21 facets. Rudin's 3-ball [31] with 14 vertices and 41 tetrahedra gives a strongly non-shellable rectilinear triangulation of a tetrahedron with all the vertices on the boundary; the vertices even can be moved slightly to yield a straight triangulation of a convex 3-polytope with 14 vertices [10]. Ziegler's ball is realizable as a straight, yet non-convex ball in 3-space. Coordinates for a rectilinear realization of Grünbaum's ball can be found in [16]. Vertex-minimal non-shellable 3-balls with 9 vertices are enumerated in [7]; see [25] for a geometric realization of one of these balls with 18 facets.

The existence of non-constructible 3-balls was shown by Lickorish [21] in 1971, but it remained unclear whether there are non-shellable 3-spheres. Non-shellable cell partitions of S^3 were first constructed by Vince [33] in 1985 and then by Armentrout [2]. In 1991, Lickorish [22] described non-shellable triangulated 3-spheres that contain a knotted triangle made of the sum of (at least) three trefoil knots. In fact, is suffices to use one single trefoil knot:

Theorem 1 (Hachimori and Ziegler [17]) *If a triangulated* 3*-ball or* 3*-sphere contains* ***any*** *knotted triangle, then it is non-constructible (and thus non-shellable). Moreover, a* 3*-ball with a knotted spanning arc consisting of at most* 2 *edges is non-constructible.*

A first explicit, but large, non-constructible triangulated 3-sphere with f-vector $f = (381, 2309, 3856, 1928)$ based on Furch's 3-ball with a knotted spanning arc consisting of one edge was constructed by Hachimori [15]. Suspensions of such spheres produce non-constructible simplicial PL d-spheres in dimensions $d \geq 3$. Examples of small non-PL (and hence non-constructible) d-spheres of dimensions $d \geq 5$ with $d+13$ vertices can be found in [5]; see also [6]. Their construction makes use of the double suspension theorem of Edwards [13] (respectively of its generalization by Cannon [9]) that double suspensions of non-spherical homology d-spheres give non-PL $(d+2)$-spheres.

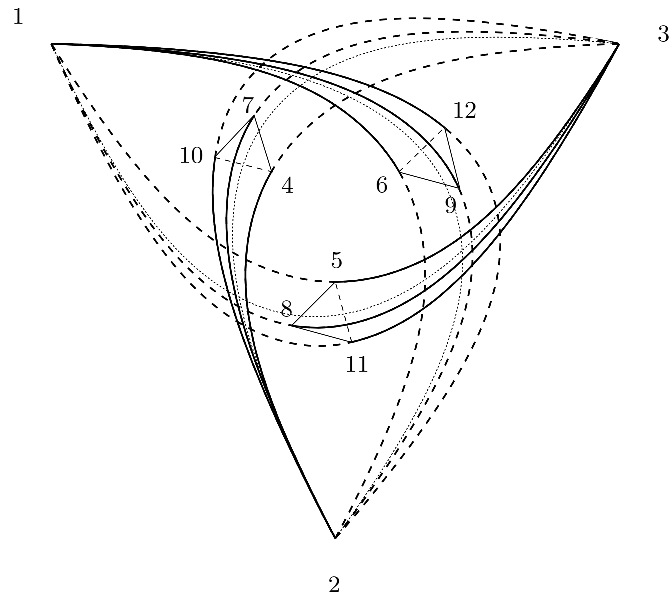

Figure 1: The trefoil knot with three protected edges.

2 The Examples

In the following, we employ the theorem of Hachimori and Ziegler to construct simplicial PL d-spheres in dimensions $d \geq 3$ with only $d + 10$ vertices that are non-constructible. From the enumeration in [7] it follows that all 3-spheres with $n \leq 10$ vertices are shellable. Hence, the non-constructible 3-sphere $S^3_{13,56}$ with 13 vertices that we are going to obtain is, if not vertex-minimal, then close to vertex-minimality.

Theorem 2 *There is a non-constructible* 3*-sphere* $S^3_{13,56}$ *with* 13 *vertices and* 56 *facets. Moreover, there are two strongly non-shellable, non-constructible* 3*-balls* $B^3_{12,37,a}$ *and* $B^3_{12,37,b}$ *with* 12 *vertices and* 37 *facets that can not be rectilinearly embedded into* $\mathbb{R}^3$.

Proof. The examples are based on a trefoil knot consisting of three edges 12, 13, and 23 (the dotted lines in Figure 1) which we embed into $\mathbb{R}^3$. We shield off the edges by enclosing every edge with three tetrahedra, as listed in the first column of Table 1. We then close the holes of the knot by gluing in the following 16 triangles:

456	146	245	356
	147	258	369
	17 10	28 11	39 12
	15 10	26 11	34 12
	45 10	56 11	46 12.

Table 1: The ball $B^3_{16,46}$.

1269	146 12	147 13	258 14	369 15	456 16
126 12	245 10	247 13	358 14	169 15	146 16
129 12	356 11	17 10 13	28 11 14	39 12 15	14 13 16
		27 10 13	38 11 14	19 12 15	24 13 16
1358		15 10 13	26 11 14	34 12 15	245 16
135 11		25 10 13	36 11 14	14 12 15	25 14 16
138 11		158 13	269 14	347 15	35 14 16
		258 13	369 14	147 15	356 16
2347					36 15 16
234 10					16 15 16
237 10					

The resulting simplicial complex C is contractible. By adding the 37 tetrahedra in the columns 2–6 of Table 1 we thicken C to a ball $B^3_{16,46}$ with 16 vertices, 46 facets, and f-vector $f = (16, 75, 106, 46)$. Since $B^3_{16,46}$ contains a trefoil knot composed of three edges, it follows from Theorem 1 of Hachimori and Ziegler that $B^3_{16,46}$ is not constructible and thus not shellable. In fact, $B^3_{16,46}$ is strongly non-shellable, as the removal of any of its facets destroys the ballness. Moreover, the presence of the 3-edge knot prevents $B^3_{16,46}$ from having a straight embedding into $\mathbb{R}^3$.

In Figure 2 we display the complex C. We also indicate the cones with respect to the vertices 13, 14, and 15 over eight of the triangles of C each, as listed in columns 3–5 of Table 1. The cone with respect to vertex 16 is then placed "above" the drawing.

The boundary of $B^3_{16,46}$ consists of 28 triangles:

1 13 16	456	45 10	56 11	46 12
2 13 16		15 10	26 11	34 12
2 14 16		15 11	26 12	34 10
3 14 16		18 11	29 12	37 10
3 15 16		28 11	39 12	17 10
1 15 16		28 13	39 14	17 15
		18 13	29 14	37 15.

If we add to $B^3_{16,46}$ the cone over these 28 triangles with respect to a new vertex 17, then we get a 3-sphere $S^3_{17,74}$ with $f = (17, 91, 148, 74)$. This 3-sphere still contains the complex C and with it the trefoil knot composed of the three edges 12, 13, and 23. Hence, $S^3_{17,74}$ is a not constructible, non-shellable sphere. By construction, $B^3_{16,46}$ and $S^3_{17,74}$ have a $\mathbb{Z}_3$-symmetry.

Since all 3-spheres with $n \leq 10$ vertices are shellable [7], 17 vertices is close to the minimal number of vertices that are needed for a non-shellable 3-sphere. In order to still improve on the number of vertices, we applied the bistellar flip program BISTELLAR [24] to $S^3_{17,74}$, under the additional restriction that the edges of the knot should not be touched. (The objective of BISTELLAR is to decrease the size of a triangulation of a manifold by performing bistellar flips that

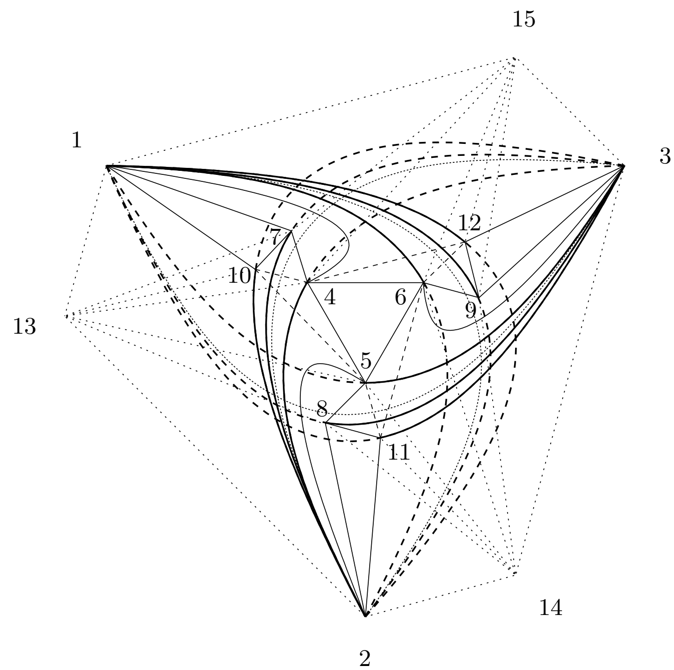

Figure 2: The contractible complex C with three cones.

locally modify the triangulation without changing the topological type; see [5] for an explicit description.) As result, we obtained a simplicial 3-sphere $S^3_{13,56}$ with $f = (13, 69, 112, 56)$. The removal of the star of vertex 13

179 13	257 13	358 13	579 13
17 11 13	258 13	359 13	6 10 11 13
19 10 13	26 11 13	36 10 13	
1 10 11 13	26 12 13	36 12 13	
	27 11 13	38 12 13	
	28 12 13	39 10 13	

from this complex yields a 12-vertex 3-ball $B^3_{12,38}$ with 38 facets, as listed in Table 2. This ball has two free facets, 2457 and 346 10, so is not strongly non-

Table 2: The ball $B^3_{12,38}$.

1269	158 10	*2457*	3467	4567
126 12	15 10 11	245 10	*346 10*	456 10
129 12	1679	258 10	359 11	5679
	167 12	269 11	367 12	569 11
1358	178 10	278 10	37 10 12	56 10 11
135 11	178 11	278 11	389 11	
138 11	17 10 12	289 11	389 12	
	19 10 12	289 12	39 10 12	
2347				
234 10				
237 10				

shellable. However, when we remove either of the two tetrahedra, we get strongly non-shellable, non-constructible 3-balls $B^3_{12,37,a}$ and $B^3_{12,37,b}$ with 37 facets and $f = (12, 58, 84, 37)$, respectively. These two balls are not isomorphic, although they have isomorphic boundaries. Both balls (and also the sphere $S^3_{13,56}$) still contain the original 3-edge trefoil knot for which, this time, the triangles

456	467	245	569
	167	258	359
	17 10	28 11	39 12
	15 10	26 11	36 12
	45 10	56 11	346

are glued in to close the holes of the knot; see Figure 3. □

Corollary 3 *For $d \geq 3$ there are non-constructible d-spheres with $d + 10$ vertices. Also there are non-constructible d-balls, $d \geq 3$, with $d + 9$ vertices and 37 facets that do not have a straight embedding into $\mathbb{R}^d$.*

Proof. The cone over a non-constructible, non-realizable d-ball is a non-constructible, non-realizable $(d+1)$-ball with the same number of facets. Similarly, the one-point suspension of a non-constructible d-sphere is a non-constructible $(d + 1)$-sphere; see [19]. □

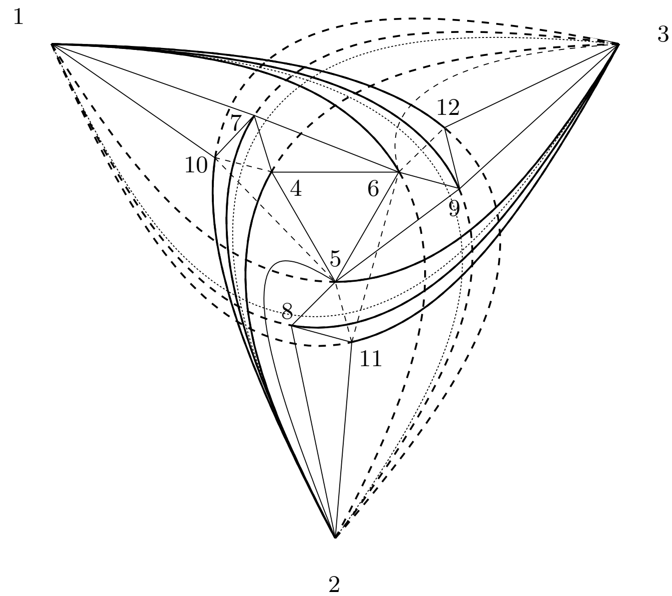

Figure 3: The 3-edge trefoil knot lying in the non-shellable sphere $S^3_{13,56}$.

Acknowledgment

The author is grateful to Günter M. Ziegler for helpful remarks.

References

[1] J. W. Alexander. The combinatorial theory of complexes. *Ann. Math.* **31**, 292–320 (1930).

[2] S. Armentrout. Knots and shellable cell partitionings of S^3. *Ill. J. Math.* **38**, 347–365 (1994).

[3] R. H. Bing. Some aspects of the topology of 3-manifolds related to the Poincaré conjecture. *Lectures on Modern Mathematics, Volume II* (T. L. Saaty, ed.), Chapter 3, 93–128. John Wiley & Sons, 1964.

[4] A. Björner. Topological methods. *Handbook of Combinatorics* (R. Graham, M. Grötschel, and L. Lovász, eds.), Chapter 34, 1819–1872. Elsevier, Amsterdam, 1995.

[5] A. Björner and F. H. Lutz. Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere. *Exp. Math.* **9**, 275–289 (2000).

[6] A. Björner and F. H. Lutz. A 16-vertex triangulation of the Poincaré homology 3-sphere and non-PL spheres with few vertices. *Electronic Geometry Model No.* 2003.04.001 (2003).

[7] J. Bokowski, D. Bremner, F. H. Lutz, and A. Martin. Combinatorial 3-manifolds with 10 vertices. In preparation.

[8] H. Bruggesser and P. Mani. Shellable decompositions of cells and spheres. *Math. Scand.* **29**, 197–205 (1971).

[9] J. W. Cannon. Shrinking cell-like decompositions of manifolds. Codimension three. *Ann. Math.* **110**, 83–112 (1979).

[10] R. Connelly and D. W. Henderson. A convex 3-complex not simplicially isomorphic to a strictly convex complex. *Math. Proc. Camb. Phil. Soc.* **88**, 299–306 (1980).

[11] G. Danaraj and V. Klee. Which spheres are shellable? *Algorithmic Aspects of Combinatorics* (B. Alspach, P. Hell, and D. J. Miller, eds.). Annals of Discrete Mathematics **2**, 33–52. North-Holland, Amsterdam, 1978.

[12] G. B. Dantzig. *Linear Programming and Extensions.* Princeton University Press, Princeton, NJ, 1963.

[13] R. D. Edwards. The double suspension of a certain homology 3-sphere is S^5. *Notices AMS* **22**, A–334 (1975).

[14] R. Furch. Zur Grundlegung der kombinatorischen Topologie. *Abh. Math. Sem. Univ. Hamburg* **3**, 69–88 (1924).

[15] M. Hachimori. A 3-sphere with a knotted triangle. (`http://infoshako.sk.tsukuba.ac.jp/~hachi/math/library/nc_sphere_eng.html`)

[16] M. Hachimori. Nonconstructible simplicial balls and a way of testing constructibility. *Discrete Comput. Geom.* **22**, 223–230 (1999).

[17] M. Hachimori and G. M. Ziegler. Decompositions of simplicial balls and spheres with knots consisting of few edges. *Math. Z.* **235**, 159–171 (2000).

[18] M. Hochster. Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes. *Ann. Math.* **96**, 318–337 (1972).

[19] M. Joswig and F. H. Lutz. Dual wedges, one-point suspensions, and wreath products. In preparation.

[20] V. Klee and P. Kleinschmidt. The d-step conjecture and its relatives. *Math. Oper. Res.* **12**, 718–755 (1987).

[21] W. B. R. Lickorish. An unsplittable triangulation. *Mich. Math. J.* **18**, 203–204 (1971).

[22] W. B. R. Lickorish. Unshellable triangulations of spheres. *Eur. J. Comb.* **12**, 527–530 (1991).

[23] E. R. Lockeberg. *Refinements in Boundary Complexes of Polytopes.* Dissertation. University College London, 1977.

[24] F. H. Lutz. GAP-program `BISTELLAR`. Version 05/02. (`http://www.math.tu-berlin.de/diskregeom/stellar/bistellar.tar.gz`)

[25] F. H. Lutz. A minimal non-shellable simplicial 3-ball with 9 vertices and 18 facets. In preparation.

[26] F. H. Lutz. Vertex-minimal not vertex-decomposable balls. In preparation.

[27] P. McMullen. The maximum numbers of faces of a convex polytope. *Mathematika* **17**, 179–184 (1970).

[28] M. H. A. Newman. On the foundations of combinatory analysis situs. I, II. *Proc. Royal Acad. Amsterdam* **29**, 611–626, 627–641 (1926).

[29] M. H. A. Newman. A property of 2-dimensional elements. *Proc. Royal Acad. Amsterdam* **29**, 1401–1405 (1926).

[30] J. S. Provan and L. J. Billera. Decompositions of simplicial complexes related to diameters of convex polyhedra. *Math. Oper. Res.* **5**, 576–594 (1980).

[31] M. E. Rudin. An unshellable triangulation of a tetrahedron. *Bull. Am. Math. Soc.* **64**, 90–91 (1958).

[32] L. Schläfli. *Theorie der vielfachen Kontinuität* (written 1850–1852). Neue Denkschriften der allgemeinen schweizerischen Gesellschaft für die Gesammten Naturwissenschaften **38**. Zürcher und Furrer, Zürich, 1901. Reprinted in: Ludwig Schläfli, 1814–1895, Gesammelte mathematische Abhandlungen, Band I, 167–387. Birkhäuser, Basel, 1950.

[33] A. Vince. A non-shellable 3-sphere. *Eur. J. Comb.* **6**, 91–100 (1985).

[34] G. M. Ziegler. *Lectures on Polytopes*. Graduate Texts in Mathematics **152**. Springer-Verlag, New York, NY, 1995. Revised edition, 1998.

[35] G. M. Ziegler. Shelling polyhedral 3-balls and 4-polytopes. *Discrete Comput. Geom.* **19**, 159–174 (1998).

Technische Universität Berlin
Fakultät II - Mathematik und Naturwissenschaften
Institut für Mathematik, Sekr. MA 6-2
Straße des 17. Juni 136
10623 Berlin
Germany
`lutz@math.tu-berlin.de`